

# APPROXIMATION OF FORWARD CURVE MODELS IN COMMODITY MARKETS WITH ARBITRAGE-FREE FINITE DIMENSIONAL MODELS

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**ABSTRACT.** In this paper we show how to approximate a Heath-Jarrow-Morton dynamics for the forward prices in commodity markets with arbitrage-free models which have a finite dimensional state space. Moreover, we recover a closed form representation of the forward price dynamics in the approximation models and derive the rate of convergence uniformly over an interval of time to maturity to the true dynamics under certain additional smoothness conditions. In the Markovian case we can strengthen the convergence to be uniform over time as well. Our results are based on the construction of a convenient Riesz basis on the state space of the term structure dynamics.

## 1. INTRODUCTION

We develop arbitrage-free approximations to the forward term structure dynamics in commodity markets. The approximating term structure models have finite dimensional state space, and therefore tractable for further analysis and numerical simulation. We provide results on the convergence of the approximating term structures and characterize the speed under reasonable smoothness properties of the true term structure. Our results are based on the construction of a convenient Riesz basis on the state space of the term structure dynamics.

In the context of fixed-income markets, Heath, Jarrow and Morton [19] propose to model the entire term structure of interest rates. Filipović [16] reinterprets this approach in the so-called Musiela parametrisation, i.e., studying the so-called forward rates as solutions of first-order stochastic partial differential equations. This class of stochastic partial differential equations is often referred to as the Heath-Jarrow-Morton-Musiela (HJMM) dynamics. This highly successful method has been transferred to other markets, including commodity and energy futures markets (see Clewlow and Strickland [14] and Benth, Saltyte Benth and Koekebakker [5]), where the term structure of forward and futures prices are modelled by similar stochastic partial differential equations.

An important stream of research in interest rate modelling has been so-called finite dimensional realizations of the solutions of the HJMM dynamics (see e.g., Björk and Svensson [12], Björk and Landen [11], Filipovic and Teichmann [18] and Tappe [24]). Starting out with an equation for the forward rates driven by a  $d$ -dimensional Wiener process, the question has been under what conditions on the volatility and drift do we get

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solutions which belongs to a finite dimensional space, that is, when can the dynamics of the whole curve be decomposed into a finite number of factors. This property has a close connection with principal component analysis (see Carmona and Tehranchi [13]), but is also convenient when it comes to further analysis like estimation, simulation, pricing and portfolio management (see Benth and Lempa [10] for the latter).

In energy markets like power and gas, there is empirical and economical evidence for high-dimensional noise. Moreover, the noise shows clear leptokurtic signs (see Benth, Šaltytė Benth and Koekebakker [5] and references therein). These empirical insights motivate the use of infinite dimensional Lévy processes driving the noise in the HJMM-dynamics modelling the forward term structure. We refer to Carmona and Tehranchi [13] for a thorough analysis of HJMM-models with infinite dimensional Gaussian noise in interest rate markets. Benth and Krühner [8] introduced a convenient class of infinite dimensional Lévy processes via subordination of Gaussian processes in infinite dimensions. These models were used in analysing stochastic partial differential equations with infinite dimensional Lévy noise in Benth and Krühner [7]. Further, pricing and hedging of derivatives in energy markets based on such models were studied in Benth and Krühner [9].

The present paper is motivated by the need of an arbitrage-free approximation of Heath, Jarrow, Morton style models – using the Musiela parametrisation – in electricity finance. Related research has been carried out by Henseler, Peters and Seydel [20] who construct a finite-dimensional affine model where a refined principle component analysis (PCA) method does yield an arbitrage free approximation of the term structure model. Our main result Theorem 4.1 states that the arbitrage-free models for the underlying forward curve process  $f(t, x)$ ,  $x \geq 0$  being time to maturity and  $t \geq 0$  is current time, can be approximated with processes of the form

$$f_k(t, x) = S_k(t) + \sum_{n=-k}^k U_n(t)g_n(x),$$

where  $S_k$  denotes the spot prices in the approximating model,  $g_{-k}, \dots, g_k$  are deterministic functions and  $U_{-k}, \dots, U_k$  are one-dimensional Ornstein Uhlenbeck type processes. Obviously, models of this type are much easier to handle in applications than general solutions for the HJMM equation. The approximation  $f_k$  is again a solution of an HJMM equation, and as such being an arbitrage-free model for the forward term structure. We prove a uniform convergence in space of  $f_k$  to the "real" forward price curve  $f$ , pointwise in time. The convergence rate is of order  $k^{-1}$  when the forward curve  $x \mapsto f(t, x)$  is twice continuously differentiable. Our approach is an alternative to numerical approximations of the HJMM dynamics based on finite difference schemes or finite element methods, where arbitrage-freeness of the approximating dynamics is not automatically ensured. We refer to Barth [1] for an analysis of finite element methods applied to stochastic partial differential equations of the type we study.

We refine our results to the Markovian case, where the convergence is slightly strengthened to be uniform over time as well. Our approach goes via the explicit construction of a Riesz basis for a subspace of the so-called Filipović space (see Filipović [16]), a separable Hilbert space of absolutely continuous functions on the positive real line with (weak) derivative disappearing at a certain speed at infinity. The basis will be the functions  $g_n$  in the approximation  $f_k$ , and the subspace is defined by concentrating the functions in

the Filipović space to a finite time horizon  $x \leq T$ . This space was defined in Benth and Krühner [7], and we extend the analysis here to accomodate the arbitrage-free finite dimensional approximation of the HJMM-dynamics. We rest on properties of  $C_0$ -semigroups and stochastic integration with respect to infinite dimensional Lévy processes (see Peszat and Zabczyk [22]) in the analysis.

This paper is organised as follows. In Section 2 we start with the mathematical formulation of the HJMM dynamics for forward rates set in the Filipović space. The Riesz basis that will make the foundation for our approximation is defined and analysed in detail in Section 3. The arbitrage-free finite dimensional approximation to term structure modelling is constructed in Section 4, where we study convergence properties. The Markovian case is analysed in the last Section 5.

## 2. THE MODEL OF THE FORWARD PRICE DYNAMICS

Throughout this paper we use the Hilbert space

$$H_\alpha := \left\{ f \in AC(\mathbb{R}_+, \mathbb{C}) : \int_0^\infty |f'(x)|^2 e^{\alpha x} dx < \infty \right\},$$

where  $AC(\mathbb{R}_+, \mathbb{C})$  denotes the space of complex-valued absolutely continuous functions on  $\mathbb{R}_+$ . We endow  $H_\alpha$  with the scalar product  $\langle f, g \rangle_\alpha := f(0)\bar{g}(0) + \int_0^\infty f'(x)\bar{g}'(x)e^{\alpha x} dx$ , and denote the associated norm by  $\|\cdot\|_\alpha$ . Filipović [16, Section 5] shows that  $(H_\alpha, \|\cdot\|_\alpha)$  is a separable Hilbert space<sup>1</sup>. This space has been used in Filipović [16] for term structure modelling of bonds and many mathematical properties have been derived therein. We will frequently refer to  $H_\alpha$  as the *Filipović space*.

We next introduce our dynamics for the term structure of forward prices in a commodity market. Denote by  $f(t, x)$  the price at time  $t$  of a forward contract where time to delivery of the underlying commodity is  $x \geq 0$ . We treat  $f$  as a stochastic process in time with values in the Filipović space  $H_\alpha$ . More specifically, we assume that the process  $\{f(t)\}_{t \geq 0}$  follows the HJM-Musiela model which we formalize next.

On a complete filtered probability space  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, P)$ , where the filtration is assumed to be complete and right continuous, we work with an  $H_\alpha$ -valued Lévy process  $\{L(t)\}_{t \geq 0}$  (cf. Peszat and Zabczyk [22, Theorem 4.27(i)] for the construction of  $H_\alpha$ -valued Lévy processes). We assume that  $L$  has finite variance and mean equal to zero, and denote its covariance operator by  $\mathcal{Q}$ . Let  $f_0 \in H_\alpha$  and  $f$  be the solution of the stochastic partial differential equation (SPDE)

$$df(t) = \partial_x f(t)dt + \beta(t)dt + \Psi(t)dL(t), \quad t \geq 0, f(0) = f_0 \quad (1)$$

where  $\beta \in L^1((\Omega \times \mathbb{R}_+, \mathcal{P}, P \otimes \lambda), H_\alpha)$ ,  $\mathcal{P}$  being the predictable  $\sigma$ -field, and  $\Psi \in \mathcal{L}_L^2(H_\alpha) := \bigcup_{T>0} \mathcal{L}_{L,T}^2(H_\alpha)$  where the latter space is defined as in Peszat and Zabczyk [22, page 113]. For  $t \geq 0$ , denote by  $\mathcal{U}_t$  the shift semigroup on  $H_\alpha$  defined by  $\mathcal{U}_t f = f(t + \cdot)$  for  $f \in \mathcal{H}_\alpha$ . It is shown in Filipović [16] that  $\{\mathcal{U}_t\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $H_\alpha$ , with generator  $\partial_x$ . Recall, that any  $C_0$ -semigroup admits the bound  $\|\mathcal{U}_t\|_{\text{op}} \leq M e^{wt}$  for some  $w, M > 0$  and any  $t \geq 0$ . Here,  $\|\cdot\|_{\text{op}}$  denotes the operator norm. In fact, in Filipović [16, Equation (5.10)] and Benth and Krühner [4, Lemma 3.4] it is shown that  $\|\mathcal{U}_t\|_{\text{op}} \leq C_{\mathcal{U}}$  for

<sup>1</sup>Note that Filipović [16] does not consider complex-valued functions. In our context, this minor extension is convenient, as will be clear later.

any  $t \geq 0$  and a constant  $C_{\mathcal{U}} := \sqrt{2(1 \wedge \alpha^{-1})}$ . Thus  $s \mapsto \mathcal{U}_{t-s}\beta(s)$  is Bochner-integrable and  $s \mapsto \mathcal{U}_{t-s}\Psi(s)$  is integrable with respect to  $L$ . The unique mild solution of (1) is

$$f(t) = \mathcal{U}_t f_0 + \int_0^t \mathcal{U}_{t-s}\beta(s) ds + \int_0^t \mathcal{U}_{t-s}\Psi(s) dL(s). \quad (2)$$

If we model the forward price dynamics  $f$  in a risk-neutral setting, the drift coefficient  $\beta(t)$  will naturally be zero in order to ensure the (local) martingale property of the process  $t \mapsto f(t, \tau - t)$ , where  $\tau \geq t$  is the time of delivery of the forward. In this case, the probability  $P$  is to be interpreted as the equivalent martingale measure (also called the pricing measure). However, with a non-zero drift, the forward model is stated under the market probability and  $\beta$  can be related to the risk premium in the market.

In energy markets like power and gas, the forward contracts deliver over a period, and forward prices can be expressed by integral operators on the Filipović space applied on  $f$  (see Benth and Krühner [3, 4] for more details).

The dynamics of  $f$  can also be considered as a model for the forward rate in fixed-income theory, see Filipović [16]. This is indeed the traditional application area and point of analysis of the SPDE in (1). Note, however, that the original no-arbitrage condition in the HJM approach for interest rate markets is different from the no-arbitrage condition used here. If  $f$  is understood as the forward rate modelled in the risk-neutral setting, there is a no-arbitrage relationship between the drift  $\beta$ , the volatility  $\sigma$  and the covariance of the driving noise  $L$ . We refer to Carmona and Tehranchi [13] for a detailed analysis.

### 3. A RIESZ BASIS FOR THE FILIPOVIĆ SPACE

In this section we introduce a Riesz basis for a suitable subspace of  $H_\alpha$  defined in Benth and Krühner [3, Appendix A] and present various of its properties. Moreover, we give refined statements for this basis and also identify new properties. We recall from Young [26] that any Riesz basis  $\{g_n\}_{n \in \mathbb{N}}$  on a separable Hilbert space can be expressed by  $g_n = \mathcal{T}e_n$  where  $\{e_n\}_{n \in \mathbb{N}}$  is an orthonormal basis and  $\mathcal{T}$  is a bounded invertible linear operator. For further properties and definitions of Riesz bases, see Young [26].

In Section 4 we want to employ the spectral method to an approximation of the SPDE in (1) involving the differential operator on the Filipović space  $H_\alpha$ . Thus, it would be convenient to have available the eigenvector basis for the differential operator. However, its eigenvectors do not seem to have nice basis properties. Instead, we propose to use a system of vectors which forms a Riesz basis which turns out to be almost an eigenvector system for the differential operator. This property will be made precise in Propositions 3.5 and 3.6. Finally, we will identify the convergence speed of the Riesz basis expansion.

Fix  $\lambda > 0$ ,  $T > 0$ , and introduce

$$\text{cut} : \mathbb{R}_+ \rightarrow [0, T), \quad x \mapsto x - \max\{Tz : z \in \mathbb{Z} : Tz \leq x\}, \quad (3)$$

and

$$\mathcal{A} : L^2([0, T], \mathbb{C}) \rightarrow L^2(\mathbb{R}_+, \mathbb{C}), \quad f \mapsto (x \mapsto e^{-\lambda x} f(\text{cut}(x))) . \quad (4)$$

Here,  $L^2(A, \mathbb{C})$  is the space of complex-valued square integrable functions on the Borel set  $A \subset \mathbb{R}_+$  equipped with the Lebesgue measure. The inner product of  $L^2(A, \mathbb{C})$  will be denoted  $(\cdot, \cdot)_2$  and the corresponding norm  $|\cdot|_2$ . We remark that the set  $A$  will be clear from the context and thus not indicated in the notation for norm and inner product.

We define

$$g_*(x) := 1, \quad (5)$$

$$g_n(x) := \frac{1}{\lambda_n \sqrt{T}} (\exp(\lambda_n x) - 1), \quad (6)$$

where

$$\lambda_n := \frac{2\pi i}{T} n - \lambda - \frac{\alpha}{2}, \quad (7)$$

for any  $n \in \mathbb{Z}$ ,  $x \geq 0$ . It is simple to verify that  $g_n \in H_\alpha$  for any  $n \in \mathbb{Z}$  and  $g_* \in H_\alpha$ . As we will see, the system of vectors  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  forms a Riesz basis and we will use this to obtain arbitrage-free finite-dimensional approximations of the forward price dynamics (1).

We start our analysis with some elementary properties of the operator  $\mathcal{A}$  which have been proven in Benth and Krühner [3].

**Lemma 3.1.**  *$\mathcal{A}$  is a bounded linear operator and its range is closed in  $L^2(\mathbb{R}_+, \mathbb{C})$ . Moreover,*

$$\frac{e^{-2T\lambda}}{1 - e^{-2T\lambda}} |f|_2^2 \leq |\mathcal{A}f|_2^2 \leq \frac{1}{1 - e^{-2T\lambda}} |f|_2^2$$

for any  $f \in L^2([0, T], \mathbb{C})$ .

*Proof.* This proof can be found in Benth and Krühner [3, Lemma A.1].  $\square$

In the following Proposition 3.3, we calculate a Riesz basis of the space  $\text{ran}(\mathcal{A})$  and its biorthogonal system. The Riesz basis will be given as the image of an orthonormal basis of  $L^2([0, T], \mathbb{C})$ . Consequently, its biorthogonal system is given by the image of  $(\mathcal{A}^{-1})^*$ , which we calculate in the Lemma below:

**Lemma 3.2.** *The dual  $(\mathcal{A}^{-1})^*$  of the inverse of  $\mathcal{A} : L^2([0, T], \mathbb{C}) \rightarrow \text{ran}(\mathcal{A})$  is given by*

$$\begin{aligned} (\mathcal{A}^{-1})^* : L^2([0, T], \mathbb{C}) &\rightarrow \text{ran}(\mathcal{A}), \\ (\mathcal{A}^{-1})^* f(x) &= (1 - e^{-2\lambda T}) e^{-\lambda x} (e^{2\lambda \text{cut}(x)} f(\text{cut}(x))) \\ &= (1 - e^{-2\lambda T}) e^{2\lambda \text{cut}(x)} \mathcal{A}f(x), \quad x \geq 0. \end{aligned}$$

*Proof.* Let  $f, g \in L^2([0, T], \mathbb{C})$  and define  $h(x) := (1 - e^{-2\lambda T}) e^{2\lambda \text{cut}(x)} \mathcal{A}f(x)$  for any  $x \geq 0$ . Then we have

$$\begin{aligned} (h, \mathcal{A}g)_2 &= \int_0^\infty h(y) \overline{\mathcal{A}g(y)} dy \\ &= (1 - e^{-2\lambda T}) \sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{2\lambda(x-nT)} (e^{-\lambda x} f(x-nT)) (e^{-\lambda x} \overline{g(x-nT)}) dx \\ &= (1 - e^{-2\lambda T}) \sum_{n=0}^\infty e^{-2\lambda nT} \int_{nT}^{(n+1)T} f(x-nT) \overline{g(x-nT)} dx \\ &= \int_0^T f(y) \overline{g(y)} dy. \end{aligned}$$

On the other hand,

$$((\mathcal{A}^{-1})^* f, \mathcal{A}g)_2 = (f, g)_2 = \int_0^T f(y) \overline{g(y)} dy.$$

Since  $g$  is arbitrary, we have  $h = (\mathcal{A}^{-1})^* f$  as claimed.  $\square$

Parts of the next proposition can be found in Benth and Krühner [3, Lemma A.3]. In that paper there appears to be a gap in the proof which we have filled here.

**Proposition 3.3.** *Define*

$$e_n(x) := \frac{1}{\sqrt{T}} \exp \left( \left( \frac{2\pi i n}{T} - \lambda \right) x \right), \quad x \geq 0, n \in \mathbb{Z}.$$

*Then  $\{e_n\}_{n \in \mathbb{Z}}$  is a Riesz basis on the closed subspace  $\text{ran}(\mathcal{A})$  of  $L^2(\mathbb{R}_+, \mathbb{C})$  and*

$$F := \{f \in L^2(\mathbb{R}_+, \mathbb{C}) : f(x) = 0, x \in [0, T]\}$$

*is a closed vector space compliment of  $\text{ran}(\mathcal{A})$ . The continuous linear projector  $\mathcal{P}_{\mathcal{A}}$  with range  $\text{ran}(\mathcal{A})$  and kernel  $F$  has operator norm  $\sqrt{\frac{1}{1-e^{-2\lambda T}}}$  and we have*

$$\mathcal{P}_{\mathcal{A}} f(x) = f(x), \quad x \in [0, T], f \in L^2(\mathbb{R}_+, \mathbb{C}).$$

*The biorthogonal system  $\{e_n\}_{n \in \mathbb{Z}}^*$  for the Riesz basis  $\{e_n\}_{n \in \mathbb{Z}}$  is given by*

$$e_n^*(x) = (1 - e^{-2\lambda T}) e^{2\lambda \text{cut}(x)} e_n(x)$$

*Proof.* Recall that the range of  $\mathcal{A}$  is a closed subspace of  $L^2(\mathbb{R}_+, \mathbb{C})$  due to the lower bound given in Lemma 3.1. Furthermore,  $\{b_n\}_{n \in \mathbb{Z}}$  with

$$b_n(x) := \frac{1}{\sqrt{T}} \exp \left( \frac{2\pi i n}{T} x \right), \quad n \in \mathbb{Z}, x \in [0, T]$$

is an orthonormal basis of  $L^2([0, T], \mathbb{C})$ . Observe, that  $e_n = \mathcal{A}b_n$  and hence  $\{e_n\}_{n \in \mathbb{Z}}$  becomes a Riesz basis of  $\text{ran}(\mathcal{A})$ .

Define the continuous linear operators

$$\mathcal{M}_{\lambda} : L^2([0, T], \mathbb{C}) \rightarrow L^2([0, T], \mathbb{C}), \mathcal{M}_{\lambda} f(x) := e^{\lambda x} f(x),$$

$$\mathcal{C} : L^2(\mathbb{R}_+, \mathbb{C}) \rightarrow L^2([0, T], \mathbb{C}), f \mapsto f|_{[0, T]}$$

and  $\mathcal{P}_{\mathcal{A}} := \mathcal{A} \mathcal{M}_{\lambda} \mathcal{C}$ . Observe, that  $\mathcal{M}_{\lambda} \mathcal{C} \mathcal{A}$  is the identity operator on  $L^2([0, T], \mathbb{C})$  and hence  $\mathcal{P}_{\mathcal{A}}^2 = \mathcal{P}_{\mathcal{A}}$ . Therefore,  $\mathcal{P}_{\mathcal{A}}$  is a continuous linear projection with kernel  $F$  and range  $\text{ran}(\mathcal{A})$ .

Let  $f \in L^2(\mathbb{R}_+, \mathbb{C})$  be orthogonal to any element of the kernel of  $\mathcal{P}_{\mathcal{A}}$ . Then  $f(x) = 0$  Lebesgue-a.e. for any  $x \geq T$ . Hence, we have

$$\begin{aligned} |\mathcal{P}_{\mathcal{A}} f|_2^2 &= \sum_{n \in \mathbb{N}} \int_{nT}^{nT+T} (e^{-\lambda x} e^{\lambda(x-nT)})^2 |f(x-nT)|^2 dx \\ &= \sum_{n \in \mathbb{N}} e^{-2n\lambda T} |f|_2^2 \\ &= \frac{1}{1 - e^{-2\lambda T}} |f|_2^2 \end{aligned}$$

and it follows that  $\|\mathcal{P}_A\|_{\text{op}} = \sqrt{\frac{1}{1-e^{-2\lambda T}}}$ .

According to Lemma 3.2, we have

$$\begin{aligned} e_n^*(x) &= (\mathcal{A}^{-1})^* b_n(x) \\ &= (1 - e^{-2\lambda T}) e^{-\lambda x} (e^{2\lambda \text{cut}(x)} b_n(\text{cut}(x))) \\ &= (1 - e^{-2\lambda T}) e^{2\lambda \text{cut}(x)} e_n(x), \end{aligned}$$

for any  $n \in \mathbb{Z}$ ,  $x \geq 0$ , as required.  $\square$

The statements collected in this section have been about the space  $L^2(\mathbb{R}_+, \mathbb{C})$  so far. However, we are actually interested in the space  $H_\alpha$  which has a natural and simple isometry to  $\mathbb{C} \times L^2(\mathbb{R}_+, \mathbb{C})$ . The next corollary will translate the  $L^2(\mathbb{R}_+, \mathbb{C})$ -statements above to  $H_\alpha$ . Before stating it, we introduce a notation for later use: Define

$$\Theta : H_\alpha \rightarrow \mathbb{C} \times L^2(\mathbb{R}_+, \mathbb{C}), f \mapsto (f(0), w_\alpha f'), \quad (8)$$

where  $w_\alpha(x) := e^{x\alpha/2}$  for  $x \geq 0$ . Then  $\Theta$  is an isometry of Hilbert spaces. Its inverse is given by

$$\Theta^{-1} : \mathbb{C} \times L^2(\mathbb{R}_+, \mathbb{C}) \rightarrow H_\alpha, (z, f) \mapsto z + \int_0^{(\cdot)} w_\alpha^{-1}(y) f(y) dy. \quad (9)$$

We use these operators to prove:

**Corollary 3.4.** *The system  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  defined in (5)-(6) is a Riesz basis of a closed subspace  $H_\alpha^T$  of  $H_\alpha$ . Indeed,  $H_\alpha^T$  is the space generated by  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$ . Moreover, there is a continuous linear projector  $\Pi$  with range  $H_\alpha^T$  and operator norm  $\sqrt{\frac{1}{1-e^{-2\lambda T}}}$  such that*

$$\Pi h(x) = h(x), \quad h \in H_\alpha, x \in [0, T].$$

Consequently,  $\Pi \mathcal{U}_t h(x) = \mathcal{U}_t \Pi h(x) = h(x+t)$  for any  $t \in [0, T]$  and any  $x \in [0, T-t]$ .

The biorthogonal system  $\{g_*^*, \{g_n^*\}_{n \in \mathbb{Z}}\}$  is given by

$$\begin{aligned} g_*^*(x) &= 1 \\ g_n^*(x) &= \int_0^x e^{-y\frac{\alpha}{2}} e_n^*(y) dy \end{aligned}$$

where  $e_n^*$  is given in Proposition 3.3 for any  $n \in \mathbb{Z}$ ,  $x \geq 0$ .

*Proof.* Let  $\{e_n\}_{n \in \mathbb{Z}}$  be the Riesz basis from Proposition 3.3,  $V$  the linear vector space generated by  $\{e_n\}_{n \in \mathbb{Z}}$  (which is in fact  $\text{ran}(\mathcal{A})$ ) and  $\mathcal{P}_A$  the projector from that proposition. Then  $\{(1, 0), \{(0, e_n)\}_{n \in \mathbb{Z}}\}$  is a Riesz basis of  $\mathbb{C} \times V$ . Furthermore,  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  is a Riesz basis of  $\Theta^{-1}(\mathbb{C} \times V)$  because  $g_* = \Theta^{-1}(1, 0)$  and  $g_n = \Theta^{-1}(0, e_n)$ . Define  $\Pi := \Theta^{-1}(\text{Id}, \mathcal{P}_A)\Theta$ . Then  $\Pi$  is a linear projector with the same bound as  $\mathcal{P}_A$  where

$$(\text{Id}, \mathcal{P}_A)(z, f) := (z, \mathcal{P}_A f), \quad z \in \mathbb{C}, f \in L^2(\mathbb{R}_+, \mathbb{C}).$$

Let  $h \in H_\alpha$ . Observe that for any  $x \in [0, T]$ ,  $\text{cut}(y) = y$  when  $0 \leq y \leq x$ . We have from the definition of the various operators that

$$\begin{aligned} \Pi h(x) &= \Theta^{-1}(\text{Id}, \mathcal{P}_A)(h(0), \exp(\alpha \cdot / 2)h')(x) \\ &= \Theta^{-1}\left((h(0), (\exp((\lambda + \alpha/2) \cdot)h')|_{[0, T]}(\text{cut}(\cdot) \exp(-\lambda \cdot)))\right)(x) \\ &= h(0) + \int_0^x e^{-(\lambda + \alpha/2)y} e^{(\lambda + \alpha/2)\text{cut}(y)} h'(\text{cut}(y)) dy \\ &= h(0) + \int_0^x h'(y) dy = h(x). \end{aligned}$$

Hence,  $\Pi h(x) = h(x)$  for any  $x \in [0, T]$ .  $\square$

We remark in passing that trivially  $g_*^* = g_*$ . In the next proposition we compute the action of the shifting semigroup  $\{\mathcal{U}_t\}_{t \geq 0}$  on the Riesz basis of Corollary 3.4 and the dual semigroup on the biorthogonal system.

**Proposition 3.5.** *For the Riesz basis  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  in (5)-(6) and its biorthogonal system  $\{g_*^*, \{g_n^*\}_{n \in \mathbb{Z}}\}$  derived in Corollary 3.4, it holds*

- (1)  $\mathcal{U}_t g_n = e^{\lambda_n t} g_n + g_n(t) g_*$  and
- (2)  $\mathcal{U}_t^* g_n^* = e^{\overline{\lambda}_n t} g_n^*$ ,

for any  $n \in \mathbb{Z}$ .

*Proof.* Claim (1) follows from a straightforward computation. For claim (2), we compute

$$\begin{aligned} \mathcal{U}_t^* g_n^* &= g_* \langle \mathcal{U}_t^* g_n^*, g_* \rangle_\alpha + \sum_{k \in \mathbb{Z}} g_k^* \langle \mathcal{U}_t^* g_n^*, g_k \rangle_\alpha \\ &= g_* \langle g_n^*, \mathcal{U}_t g_* \rangle_\alpha + \sum_{k \in \mathbb{Z}} g_k^* \langle g_n^*, \mathcal{U}_t g_k \rangle_\alpha \\ &= e^{\overline{\lambda}_n t} g_n^* \end{aligned}$$

for any  $n \in \mathbb{Z}$ ,  $t \geq 0$ . Thus, the Proposition follows.  $\square$

A certain Lie commutator plays a crucial role in comparing projected solutions to the SPDE (1) with solutions to the approximation. In the next proposition, we derive the essential results for convergence which will be used in the next Section to analyse approximations of the SPDE (1).

**Proposition 3.6.** *Let  $k \in \mathbb{N}$ ,  $t \geq 0$ ,  $H_\alpha^T$  be the closed subspace of  $H_\alpha$  generated by the Riesz basis  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  defined in (5)-(6) with biorthogonal system  $\{g_*^*, \{g_n^*\}_{n \in \mathbb{Z}}\}$  given in Corollary 3.4. Define the projection*

$$\Pi_k : H_\alpha^T \rightarrow \text{span}\{g_*, g_{-k}, \dots, g_k\}, h \mapsto h(0)g_* + \sum_{n=-k}^k g_n \langle h, g_n^* \rangle_\alpha,$$

$c_{k,t} := \sum_{|n| > k} g_n(t) g_n^*$  and the operator

$$\mathcal{C}_{k,t} : H_\alpha^T \rightarrow \text{span}\{g_*^*\}, h \mapsto \langle h, c_{k,t} \rangle_\alpha g_*^*.$$



Then,  $\|\Pi_k\|_{op}$  is bounded uniformly in  $k$ ,  $\Pi_k h \rightarrow h$ ,  $\sup_{s \in [0, t]} \|\mathcal{C}_{k, s} h\|_\alpha \rightarrow 0$  for  $k \rightarrow \infty$  and any  $h \in H_\alpha^T$ , and  $[\Pi_k, \mathcal{U}_t] = \mathcal{C}_{k, t}$ . Here,  $[\Pi_k, \mathcal{U}_t]$  denotes the Lie commutator of  $\Pi_k$  and  $\mathcal{U}_t$ , that is  $[\Pi_k, \mathcal{U}_t] = \Pi_k \mathcal{U}_t - \mathcal{U}_t \Pi_k$ .

Moreover, let  $X$  be a stochastic process with values in  $H_\alpha^T$  such that  $X(t) = Y(t) + M(t)$  for some square integrable process  $Y$  of finite variation and a square integrable martingale  $M$ . Then,

$$\lim_{k \rightarrow \infty} \int_0^t \mathcal{C}_{k, t-s} dX(s) = 0,$$

where the convergence is in  $L^2(\Omega, H_\alpha)$ .<sup>2</sup>

*Proof.* Let  $h \in H_\alpha^T$ . Since  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  is a Riesz basis of  $H_\alpha^T$  we have

$$h = g_* \langle h, g_* \rangle_\alpha + \sum_{n \in \mathbb{Z}} g_n \langle h, g_n^* \rangle_\alpha,$$

and hence we get  $\Pi_k h \rightarrow h$  for  $k \rightarrow \infty$ .

We prove that the operator norm of  $\Pi_k$  is uniformly bounded in  $k \in \mathbb{N}$ . Recall from Corollary 3.4 and (9)  $g_n = \Theta^{-1}(0, \mathcal{A}b_n)$ ,  $n \in \mathbb{Z}$  and  $g_* = \Theta^{-1}(1, 0)$ , where  $\mathcal{A}$  is defined in (4) and  $\{b_n\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $L^2([0, T], \mathbb{C})$ . Without loss of generality, we assume  $h(0) = 0$  for  $h \in H_\alpha^T$ , and find that

$$\Pi_k h = \sum_{n=-k}^k g_n \langle h, g_n^* \rangle_\alpha = \sum_{n=-k}^k \mathcal{T} b_n (\mathcal{T}^{-1} h, b_n)_2 = \mathcal{T} \sum_{n=-k}^k b_n (\mathcal{T}^{-1} h, b_n)_2.$$

Here,  $\mathcal{T}f := \Theta^{-1}(0, \mathcal{A}f) \in H_\alpha$  for  $f \in L^2([0, T], \mathbb{C})$ , which is a bounded linear operator. Hence, since  $\sum_{n=-k}^k b_n (\mathcal{T}^{-1} h, b_n)_2$  is the projection of  $\mathcal{T}^{-1} h \in L^2([0, T], \mathbb{C})$  down to its first  $2k + 1$  coordinates,

$$\|\Pi_k h\|_\alpha \leq \|\mathcal{T}\|_{op} \left\| \sum_{n=-k}^k b_n (\mathcal{T}^{-1} h, b_n)_2 \right\|_2 \leq \|\mathcal{T}\|_{op} \|\mathcal{T}^{-1} h\|_2$$

But since  $\mathcal{T}^{-1}$  also is a bounded operator, it follows that  $\|\Pi_k\|_{op} \leq \|\mathcal{T}\|_{op} \|\mathcal{T}^{-1}\|_{op}$ .

Benth and Krühner [3, Lemma 3.2] yields that convergence in  $H_\alpha$  implies local uniform convergence. Thus, as we know  $h - \Pi_k h \rightarrow 0$ , it holds

$$\sup_{s \in [0, t]} |h(s) - \Pi_k h(s)| \rightarrow 0,$$

for  $k \rightarrow \infty$ . Hence, we find

$$\sup_{s \in [0, t]} \left\| \sum_{|n| > k} g_n(s) \langle h, g_n^* \rangle_\alpha \right\|_\alpha = \sup_{s \in [0, t]} |h(s) - \Pi_k h(s)| \rightarrow 0,$$

for  $k \rightarrow \infty$ . Therefore,  $\sup_{s \in [0, t]} \|\mathcal{C}_{k, s} h\|_\alpha \rightarrow 0$  for  $k \rightarrow \infty$ .

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<sup>2</sup> $L^2(\Omega, H_\alpha)$  denotes the space of  $H_\alpha$ -valued random variables  $Z$  with  $\mathbb{E}[\|Z\|_\alpha^2] < \infty$ .

Let  $n \in \mathbb{Z}$ . Then, by Proposition 3.5

$$\begin{aligned} [\Pi_k, \mathcal{U}_t]g_n &= \Pi_k(e^{\lambda_n t} g_n + g_n(t)g_*) - 1_{\{|n| \leq k\}} \mathcal{U}_t g_n \\ &= 1_{\{|n| \leq k\}} e^{\lambda_n t} g_n + g_n(t)g_* - 1_{\{|n| \leq k\}} (e^{\lambda_n t} g_n + g_n(t)g_*) \\ &= 1_{\{|n| > k\}} g_n(t)g_* \\ &= \mathcal{C}_{k,t} g_n \end{aligned}$$

for any  $t \geq 0$ . Moreover,

$$[\Pi_k, \mathcal{U}_t]g_* = \Pi_k g_* - \mathcal{U}_t g_* = 0 = \mathcal{C}_{k,t} g_*.$$

Let  $\langle\langle M, M \rangle\rangle(t) = \int_0^t Q_s d\langle M, M \rangle(s)$  be the quadratic variation processes of the martingale  $M$  given in Peszat and Zabczyk [22, Theorem 8.2]<sup>3</sup>. Then, Peszat and Zabczyk [22, Theorem 8.7(ii)] yields

$$\mathbb{E} \left( \left\| \int_0^t \mathcal{C}_{k,t-s} dM(s) \right\|_\alpha^2 \right) = \mathbb{E} \int_0^t \text{Tr}(\mathcal{C}_{k,t-s} Q_s \mathcal{C}_{k,t-s}^*) d\langle M, M \rangle(s).$$

Recall that for  $h \in H_\alpha^T$ , we find  $\mathcal{C}_{k,t} h = \langle h, c_{k,t} \rangle_\alpha g_*$ . Thus,

$$\langle h, \mathcal{C}_{k,t}^* g_* \rangle_\alpha = \langle \mathcal{C}_{k,t} h, g_* \rangle_\alpha = \langle h, c_{k,t} \rangle_\alpha,$$

which gives that  $\mathcal{C}_{k,t}^* g_* = c_{k,t}$ . For  $g \in H_\alpha^T$  orthogonal to  $g_*$  we have

$$\langle h, \mathcal{C}_{k,t}^* g \rangle_\alpha = \langle \mathcal{C}_{k,t} h, g \rangle_\alpha = \langle h, c_{k,t} \rangle_\alpha \langle g_*, g \rangle_\alpha = 0$$

for any  $h \in H_\alpha^T$  and hence  $\mathcal{C}_{k,t}^* g = 0$ . We get

$$\begin{aligned} \text{Tr}(\mathcal{C}_{k,t-s} Q_s \mathcal{C}_{k,t-s}^*) &= \langle \mathcal{C}_{k,t-s} Q_s \mathcal{C}_{k,t-s}^* g_*, g_* \rangle_\alpha \\ &= \langle Q_s c_{k,t-s}, c_{k,t-s} \rangle_\alpha \\ &\leq \|c_{k,t-s}\|_\alpha^2 \text{Tr}(Q_s). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E} \left( \left\| \int_0^t \mathcal{C}_{k,t-s} dM(s) \right\|_\alpha^2 \right) &= \mathbb{E} \int_0^t \text{Tr}(\mathcal{C}_{k,t-s} Q_s \mathcal{C}_{k,t-s}^*) d\langle M, M \rangle(s) \\ &\leq \sup_{s \in [0,t]} \|c_{k,s}\|_\alpha^2 \mathbb{E} \left( \int_0^t \text{Tr}(Q_s) d\langle M, M \rangle(s) \right) \\ &= \sup_{s \in [0,t]} \|c_{k,s}\|_\alpha^2 \mathbb{E} (\|M(t) - M(0)\|_\alpha^2) \\ &\rightarrow 0 \end{aligned}$$

for  $k \rightarrow \infty$ . Similarly, we get

$$\left\| \int_0^t \mathcal{C}_{k,t-s} dY(s) \right\|_\alpha^2 \leq \sup_{s \in [0,t]} \|c_{k,s}\|_\alpha^2 \left( \int_0^t \|dY\|_\alpha(s) \right)^2 \rightarrow 0$$

as  $k \rightarrow 0$ , where  $\|dY\|_\alpha$  denotes the total variation measure associated with  $dY$  (see Dinculeanu [15, Definition §2.1]). The claim follows.  $\square$

<sup>3</sup>In Peszat and Zabczyk [22],  $\langle\langle \cdot, \cdot \rangle\rangle$  is called the operator angle bracket process, while  $\langle \cdot, \cdot \rangle$  is the angle bracket process.

The projection operator  $\Pi_k$  plays an important role in the arbitrage-free approximation of the forward term structure. For notational convenience, we denote

$$H_\alpha^{T,k} := \text{span}\{g_*, g_{-k}, \dots, g_k\}, \quad (10)$$

for any  $k \in \mathbb{N}$ . From the above considerations, we have that  $\Pi_k$  projects the space  $H_\alpha^T$  down to  $H_\alpha^{T,k}$ .

Our next aim is to identify the convergence speed of approximations in  $H_\alpha^{T,k}$  of certain smooth elements  $f \in H_\alpha^T$ , that is, how close is  $\Pi_k f$  to  $f$  in terms of number of Riesz basis functions. We show a couple of technical results first.

**Corollary 3.7.** *Let  $f \in H_\alpha^T$ . Then, we have*

$$\frac{e^{-2\lambda T}}{1 - e^{-2\lambda T}} \left( |f(0)|^2 + \sum_{n \in \mathbb{Z}} |\langle f, g_n^* \rangle_\alpha|^2 \right) \leq \|f\|_\alpha^2 \leq \frac{1}{1 - e^{-2\lambda T}} \left( |f(0)|^2 + \sum_{n \in \mathbb{Z}} |\langle f, g_n^* \rangle_\alpha|^2 \right).$$

*Proof.* Corollary 3.4 states that  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  is a Riesz basis of  $H_\alpha^T$ . Moreover, it is given by  $g_* = \Theta^{-1}(1, 0)$ ,  $g_n = \Theta^{-1}(0, e_n)$  for any  $n \in \mathbb{Z}$  where  $\Theta$  is the isometry given in (9) and  $\{e_n\}_{n \in \mathbb{Z}}$  is the Riesz basis given in Proposition 3.3. Moreover, Lemma 3.1 yields that  $e_n = \mathcal{A}b_n$  for any  $n \in \mathbb{Z}$  where  $\{b_n\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $L^2([0, T], \mathbb{C})$  and  $\|\mathcal{A}\|_{\text{op}}^2 \leq \frac{1}{1 - e^{-2\lambda T}}$ . Thus, we can construct a Hilbert space with orthonormal basis  $\{b_*, \{b_n\}_{n \in \mathbb{Z}}\}$  and a bounded linear operator  $\mathcal{B}$  with  $\|\mathcal{B}\|_{\text{op}}^2 \leq \frac{1}{1 - e^{-2\lambda T}}$ , such that  $g_* = \mathcal{B}b_*$ ,  $g_n = \mathcal{B}b_n$ . Thus, we have

$$\begin{aligned} \|f\|_\alpha^2 &= \|g_* \langle f, g_* \rangle_\alpha + \sum_{n \in \mathbb{Z}} g_n \langle f, g_n^* \rangle_\alpha\|_\alpha^2 \\ &= \|\mathcal{B}b_* \langle f, g_* \rangle_\alpha + \sum_{n \in \mathbb{Z}} \mathcal{B}b_n \langle f, g_n^* \rangle_\alpha\|_\alpha^2 \\ &\leq \frac{1}{1 - e^{-2\lambda T}} \left( |\langle f, g_* \rangle_\alpha|^2 + \sum_{n \in \mathbb{Z}} |\langle f, g_n^* \rangle_\alpha|^2 \right) \end{aligned}$$

where  $\{g_*, \{g_n^*\}_{n \in \mathbb{Z}}\}$  denotes the biorthogonal system to  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  given in Corollary 3.4. The lower inequality simply uses the lower inequality of Lemma 3.1 instead.  $\square$

The next technical result connects the inner product of elements in  $H_\alpha^T$  with the biorthogonal basis functions to a simple Fourier-like integral on  $[0, T]$ :

**Corollary 3.8.** *Assume  $f \in H_\alpha^T$ . Then, for any  $n \in \mathbb{Z}$ ,*

$$\langle f, g_n^* \rangle_\alpha = (1 - e^{-2\lambda T})^{-1} T^{-1/2} \int_0^T f'(x) \exp \left( \left( -\frac{2\pi i}{T} n - \lambda + \frac{\alpha}{2} \right) x \right) dx$$

*Proof.* First, recall that  $g_n^* = \Theta^*(0, e_n)$  for  $n \in \mathbb{Z}$ , where  $\Theta$  is defined in the (9). Thus,

$$\begin{aligned} \langle f, g_n^* \rangle &= \langle f, \Theta^*(0, e_n) \rangle_\alpha \\ &= (\Theta f, (0, e_n))_{\mathbb{C} \times L^2(\mathbb{R}_+)} \\ &= ((f(0), e^{\alpha/2} f'), (0, e_n))_{\mathbb{C} \times L^2(\mathbb{R}_+)} \\ &= (e^{\alpha/2} f', e_n)_2. \end{aligned}$$

Note that  $\exp(\alpha \cdot /2)f'$  and  $e_n = \mathcal{A}b_n$  are elements of  $\text{ran}(\mathcal{A})$ . If  $h \in \text{ran}(\mathcal{A})$ , then there exists a  $\hat{h} \in L^2([0, T], \mathbb{C})$  such that  $h = \mathcal{A}\hat{h}$ , or,  $h(x) = \exp(-\lambda x)\hat{h}(\text{cut}(x))$ . Observe that for  $x \in [0, T]$ ,  $\hat{h}(x) = \exp(\lambda x)h(x)$ . Then, if  $g \in \text{ran}(\mathcal{A})$ , we find

$$\begin{aligned}
(h, g)_2 &= \int_0^\infty h(x)\overline{g(x)} dx \\
&= \int_0^\infty e^{-2\lambda x}\hat{h}(\text{cut}(x))\overline{\hat{g}(\text{cut}(x))} dx \\
&= \sum_{n=0}^\infty e^{-2\lambda nT} \int_{nT}^{(n+1)T} e^{-2\lambda(x-nT)}\hat{h}(\text{cut}(x))\overline{\hat{g}(\text{cut}(x))} dx \\
&= \sum_{n=0}^\infty e^{-2\lambda nT} \int_0^T e^{-2\lambda x}\hat{h}(x)\overline{\hat{g}(x)} dx \\
&= (1 - e^{-2\lambda T})^{-1} \int_0^T h(x)\overline{g(x)} dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle f, g_n^* \rangle &= (1 - e^{-2\lambda T})^{-1} \int_0^T e^{\alpha x/2} f'(x) \overline{e_n(x)} dx \\
&= (1 - e^{-2\lambda T})^{-1} T^{-1/2} \int_0^T f'(x) \exp\left(\left(-\frac{2\pi i}{T}n - \lambda + \frac{\alpha}{2}\right)x\right) dx
\end{aligned}$$

Hence, the result follows.  $\square$

With this results at hand, we can prove a convergence rate of order  $1/k$  for sufficiently smooth functions in  $H_\alpha^T$ .

**Proposition 3.9.** *Assume  $f \in H_\alpha^T$  is such that  $f|_{[0, T]}$  is twice continuously differentiable. Then, we have*

$$\|f - \Pi_k f\|_\alpha^2 \leq \frac{C_1}{k},$$

for any  $k \in \mathbb{N}$ , where

$$C_1 = \frac{T \left| f'(T)e^{T(-\lambda+\alpha/2)} - f'(0) \right|^2 + \left( \int_0^T |f''(x)|e^{x(-\lambda+\alpha/2)} dx \right)^2}{\pi^2(1 - e^{-2\lambda T})^3},$$

and we recall the projection operator  $\Pi_k$  from Proposition 3.6.

*Proof.* Corollary 3.7 yields

$$\|f - \Pi_k f\|_\alpha^2 = \left\| \sum_{|n|>k} g_n \langle f, g_n^* \rangle_\alpha \right\|_\alpha^2 \leq C \sum_{|n|>k} |\langle f, g_n^* \rangle_\alpha|^2$$

where  $C := (1 - e^{-2\lambda T})^{-1}$ . Define  $h_n(x) := \exp(\xi_n x)$ ,  $x \geq 0$ , where we denote  $\xi_n = -\frac{2\pi i}{T}n - \lambda + \frac{\alpha}{2}$ . Then, by Corollary 3.8 and integration-by-parts we find

$$\begin{aligned} |\langle f, g_n^* \rangle_\alpha|^2 &= C^2 T^{-1} \left| \int_0^T f'(x) h_n(x) dx \right|^2 \\ &= C^2 T^{-1} \frac{1}{|\xi_n|^2} \left| f'(T) h_n(T) - f'(0) h_n(0) - \int_0^T f''(x) h_n(x) dx \right|^2 \\ &\leq \frac{2C^2}{T} \frac{1}{|\xi_n|^2} A_f, \end{aligned}$$

for any  $n \in \mathbb{Z} \setminus \{0\}$ , where the constant  $A_f$  is

$$A_f := |f'(T) e^{T(-\lambda+\alpha/2)} - f'(0)|^2 + \left( \int_0^T |f''(x) e^{x(\lambda-\alpha/2)} dx \right)^2.$$

Moreover, we have

$$\sum_{|n|>k} \frac{1}{|\xi_n|^2} = 2 \sum_{n>k} \frac{1}{|\xi_n|^2} \leq \frac{T^2}{2\pi^2 k}.$$

Putting the estimates together, we get

$$\|f - \Pi_k f\|_\alpha^2 \leq A_f \frac{C^3 T}{\pi^2 k},$$

as claimed.  $\square$

We can find a similar convergence rate for  $c_{k,t}$ , a result which becomes useful later:

**Lemma 3.10.** *Let  $c_{k,t}$  be given as in Proposition 3.6. Then,*

$$\|c_{k,t}\|_\alpha^2 \leq \frac{C_2}{k},$$

for any  $k \in \mathbb{N}$ , where  $C_2 = T/\pi^2(1 - \exp(-2\lambda T))$ .

*Proof.* We appeal to Corollary 3.7, using  $\{g_n^*\}_{n \in \mathbb{Z}}$  as the Riesz basis with biorthogonal system  $\{g_n\}_{n \in \mathbb{Z}}$ , to find

$$\begin{aligned} \|c_{k,t}\|_\alpha^2 &= \left\| \sum_{|n|>k} g_n(t) g_n^* \right\|_\alpha^2 \\ &\leq C \sum_{|n|>k} |g_n(t)|^2 \\ &= \frac{C}{T} \sum_{|n|>k} \frac{1}{|\lambda_n|^2} |e^{\lambda_n t} - 1|^2 \\ &\leq \frac{2C}{T} (1 + e^{-(2\lambda+\alpha)t}) \sum_{|n|>k} \frac{1}{|\lambda_n|^2} \\ &\leq \frac{CT}{\pi^2} \frac{1}{k}, \end{aligned}$$

for  $C = (1 - \exp(-2\lambda T))^{-1}$ . Hence, the result follows.  $\square$

With these results we are now in the position to study arbitrage-free approximations of the forward dynamics in (1).

#### 4. ARBITRAGE FREE APPROXIMATION OF FORWARD TERM STRUCTURE MODELS

In this section we find an arbitrage-free approximation of a forward term structure model – stated in the Heath-Jarrow-Morton-type setup – which lives in a finite dimensional state space. We furthermore derive the convergence speed of the approximation, and extend the results to account for forward contracts delivering the underlying commodity over a period which is the case for electricity and gas.

Consider the SPDE (1) with a mild solution  $f \in H_\alpha$  given by (2). We recall from (5)-(6) and Corollary 3.4 the Riesz basis  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}$  on the space  $H_\alpha^T$  with the biorthogonal system  $\{g_*, \{g_n^*\}_{n \in \mathbb{Z}}\}$ . Furthermore,  $\Pi$  is the projection of  $H_\alpha$  on  $H_\alpha^T$ , while from (10) and Proposition 3.5 we have the projection  $\Pi_k$  of  $H_\alpha^T$  on  $H_\alpha^{T,k}$  and the operator  $\mathcal{C}_{k,t}$  for  $k \in \mathbb{N}$ ,  $t \geq 0$ . Let us define the continuous linear operator  $\Lambda_k : H_\alpha \rightarrow H_\alpha^{T,k}$  by

$$\Lambda_k = \Pi_k \Pi \quad (11)$$

for any  $k \in \mathbb{N}$ . The following theorem is one of the main results of the paper:

**Theorem 4.1.** *For  $k \in \mathbb{N}$ , let  $f_k$  be the mild solution of the SPDE*

$$df_k(t) = \partial_x f_k(t) dt + \Lambda_k \beta(t) dt + \Lambda_k \Psi(t) dL(t), \quad t \geq 0, f_k(0) = \Lambda_k f_0. \quad (12)$$

*Then, we have*

- (1)  $\mathbb{E} \left[ \sup_{x \in [0, T-t]} |f_k(t, x) - f(t, x)|^2 \right] \rightarrow 0$  for  $k \rightarrow \infty$  and any  $t \in [0, T]$ ,
- (2)  $f_k$  takes values in the finite dimensional space  $H_\alpha^{T,k}$ , moreover,  $f_k$  is a strong solution to the SPDE (12), i.e.  $f_k \in \text{dom}(\partial_x)$ ,  $t \mapsto \partial_x f_k(t)$  is  $P$ -a.s. Bochner-integrable and

$$f_k(t) = f_k(0) + \int_0^t (\partial_x f_k(s) + \Lambda_k \beta(s)) ds + \int_0^t \Lambda_k \Psi(s) dL(s),$$

(3) and,

$$f_k(t) = S_k(t) + \sum_{n=-k}^k \left( e^{\lambda_n t} \langle f_k(0), g_n^* \rangle_\alpha + \int_0^t e^{\lambda_n(t-s)} dX_n(s) \right) g_n,$$

where  $S_k(t) = \delta_0(f_k(t))$  and  $X_n(t) := \int_0^t \langle \Pi \beta(s) ds + \Pi \Psi(s) dL(s), g_n^* \rangle_\alpha$  for any  $n \in \mathbb{Z}$ ,  $t \geq 0$ .

*Proof.* (1) Define

$$f_\Pi(t) := \mathcal{U}_t \Pi f_0 + \int_0^t \mathcal{U}_{t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)), \quad t \geq 0.$$

Since  $f_k$  is a mild solution, we have

$$\begin{aligned}
f_k(t) &= \mathcal{U}_t \Pi_k \Pi f_0 + \int_0^t \mathcal{U}_{t-s} \Pi_k (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)) \\
&= \Pi_k \mathcal{U}_t \Pi f_0 + \int_0^t \Pi_k \mathcal{U}_{t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)) \\
&\quad - \mathcal{C}_{k,t} \Pi f_0 - \int_0^t \mathcal{C}_{k,t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)) \\
&= \Pi_k \left( \mathcal{U}_t \Pi f_0 + \int_0^t \mathcal{U}_{t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)) \right) \\
&\quad - \mathcal{C}_{k,t} \Pi f_0 - \int_0^t \mathcal{C}_{k,t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)) \\
&= \Pi_k(f_\Pi(t)) - \mathcal{C}_{k,t} \Pi f_0 - \int_0^t \mathcal{C}_{k,t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s))
\end{aligned}$$

for any  $t \geq 0$ . From Benth and Krühner [3, Lemma 3.2] the sup-norm is dominated by the  $H_\alpha$ -norm. Thus, there is a constant  $c > 0$  such that

$$\mathbb{E} \left[ \sup_{x \in [0, T-t]} |\Pi_k(f_\Pi(t, x)) - f_\Pi(t, x)|^2 \right] \leq c \mathbb{E} [\|(\Pi_k - \mathcal{I})f_\Pi(t)\|_\alpha^2]$$

for any  $t \geq 0$  where  $\mathcal{I}$  denotes the identity operator on  $H_\alpha$ . The dominated convergence theorem yields that the right-hand side converges to 0 for  $k \rightarrow \infty$ . Clearly, we have

$$\sup_{x \in [0, T-t]} |\mathcal{C}_{k,t} f_\Pi(0, x)| \leq c \|\mathcal{C}_{k,t} f_\Pi(0)\|_\alpha \rightarrow 0,$$

for  $k \rightarrow \infty$ . Proposition 3.6 states that

$$\mathbb{E} \left\| \int_0^t \mathcal{C}_{k,t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)) \right\|_\alpha^2 \rightarrow 0,$$

for  $k \rightarrow 0$ . Hence, we have

$$\mathbb{E} \left( \sup_{x \in [0, T-t]} |f_k(t, x) - f_\Pi(t, x)|^2 \right) \rightarrow 0,$$

for  $k \rightarrow \infty$  and any  $t \in [0, T]$ . Since  $f_\Pi(t, x) = f(t, x)$  for any  $t \in [0, T]$ ,  $x \in [0, T-t]$  the first part follows.

(2) Note first that  $\partial_x g_n(x) = \exp(\lambda_n x) / \sqrt{T} = \lambda_n g_n(x) + g_*(x) / \sqrt{T}$ , and hence  $\partial_x g_n \in H_\alpha^{T,k}$  whenever  $|n| \leq k$ . Thus,  $H_\alpha^{T,k}$  is invariant under the generator  $\partial_x$ , and its restriction to  $H_\alpha^{T,k}$  is continuous and bounded. We find that  $f_k$  takes values only in  $H_\alpha^{T,k}$  because

$$\begin{aligned}
f_k(t) &= \Pi_k \left( \mathcal{U}_t \Pi f_0 + \int_0^t \mathcal{U}_{t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)) \right) \\
&\quad - \mathcal{C}_{k,t} \Pi f_0 - \int_0^t \mathcal{C}_{k,t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)),
\end{aligned}$$

where all summands are clearly in  $H_\alpha^{T,k}$ .

(3) As  $f_k(t) \in H_\alpha^{T,k}$ , we have the representation

$$f_k(t) = \langle f_k(t), g_*^* \rangle_\alpha g_* + \sum_{n=-k}^k \langle f_k(t), g_n^* \rangle_\alpha g_n.$$

Since  $g_*^* = 1$ , we find that  $\langle f_k(t), g_*^* \rangle_\alpha = f_k(t, 0) = \delta_0(f_k(t))$ . Thus, from the mild solution of (12) we find, using Proposition 3.5

$$\begin{aligned} f_k(t) &= S_k(t) + \sum_{n=-k}^k \left\langle \mathcal{U}_t f_k(0) + \int_0^t \mathcal{U}_{t-s} (\Lambda_k \beta(s) ds + \Lambda_k \Psi(s) dL(s)), g_n^* \right\rangle_\alpha g_n \\ &= S_k(t) + \sum_{n=-k}^k \langle f_k(0), \mathcal{U}_t^* g_n^* \rangle_\alpha g_n \\ &\quad + \sum_{n=-k}^k \int_0^t \langle \Lambda_k \beta(s) ds + \Lambda_k \Psi(s) dL(s), \mathcal{U}_{t-s}^* g_n^* \rangle_\alpha g_n \\ &= S_k(t) + \sum_{n=-k}^k e^{\lambda_n t} \langle f_k(0), g_n^* \rangle_\alpha g_n \\ &\quad + \sum_{n=-k}^k \int_0^t e^{\lambda_n(t-s)} \langle \Lambda_k \beta(s) ds + \Lambda_k \Psi(s) dL(s), g_n^* \rangle_\alpha g_n. \end{aligned}$$

Observe that for any  $f \in H_\alpha$ ,

$$\Lambda_k f = \Pi_k(\Pi f) = (\Pi f)(0) g_* + \sum_{m=-k}^k \langle \Pi f, g_m^* \rangle_\alpha g_m,$$

and since  $\{g_*, \{g_n\}_{n \in \mathbb{Z}}\}, \{g_*^*, \{g_n^*\}_{n \in \mathbb{Z}}\}$  are biorthogonal systems

$$\langle \Lambda_k f, g_n^* \rangle_\alpha = (\Pi f)(0) \langle g_*, g_n^* \rangle_\alpha + \sum_{m=-k}^k \langle \Pi f, g_m^* \rangle_\alpha \langle g_m, g_n^* \rangle_\alpha = \langle \Pi f, g_n^* \rangle_\alpha 1_{\{|n| \leq k\}}.$$

Hence, the claim follows.  $\square$

Another view on Theorem 4.1 is that all processes in the  $k$ -th approximation of  $f$  can be expressed in terms of the factor processes  $X_*, X_{-k}, \dots, X_k$ , as stated below.

**Corollary 4.2.** *Under the assumptions and notations of Theorem 4.1, we have for  $k \in \mathbb{N}$ ,*

$$f_k(t, x) = S_k(t) + \sum_{n=-k}^k U_n(t) g_n(x),$$

for any  $0 \leq t < \infty$  and  $x \geq 0$ . Here,

$$S_k(t) = S_k(0) + X_*(t) + \sum_{n=-k}^k \left( g_n(t) U_n(0) + \int_0^t g_n(t-s) dX_n(s) \right),$$



with,

$$\begin{aligned} X_n(t) &:= \left\langle \int_0^t (\Pi\beta(s)ds + \Pi\Psi(s)dL(s)), g_n^* \right\rangle_\alpha, \\ X_*(t) &:= \left\langle \int_0^t (\Pi\beta(s)ds + \Pi\Psi(s)dL(s)), g_* \right\rangle_\alpha, \\ U_n(t) &:= e^{\lambda_n t} \langle f_k(0), g_n^* \rangle + \int_0^t e^{\lambda_n(t-s)} dX_n(s) \end{aligned}$$

for  $n \in \{-k, \dots, k\}$ .

*Proof.* The first equation is a restatement of (3) in Theorem 4.1. Proposition 3.5 yields

$$\langle \mathcal{U}_t h, g_* \rangle_\alpha = \langle h, g_* \rangle_\alpha + \sum_{n=-k}^k g_n(t) \langle h, g_n^* \rangle_\alpha$$

for any  $h \in H_\alpha^{T,k}$  with  $h = \langle h, g_* \rangle_\alpha g_* + \sum_{n=-k}^k \langle h, g_n^* \rangle_\alpha g_n$ . Thus, since  $g_* = 1$  and  $g_n(0) = 0$  we have

$$\begin{aligned} S_k(t) &= f_k(t, 0) \\ &= \langle f_k(t), g_* \rangle_\alpha \\ &= \langle \mathcal{U}_t f_k(0), g_* \rangle_\alpha + \int_0^t \langle \mathcal{U}_{t-s} (\Lambda_k \beta(s) ds + \Lambda_k \Psi(s) dL(s)), g_* \rangle_\alpha \\ &= \langle f_k(0), g_* \rangle_\alpha + \sum_{n=-k}^k g_n(t) \langle f_k(0), g_n^* \rangle_\alpha \\ &\quad + \int_0^t \langle \Lambda_k \beta(s) ds + \Lambda_k \Psi(s) dL(s), g_* \rangle_\alpha \\ &\quad + \sum_{n=-k}^k \int_0^t g_n(t-s) \langle \Lambda_k \beta(s) + \Lambda_k \Psi(s) dL(s), g_n^* \rangle_\alpha. \end{aligned}$$

As in the proof of Theorem 4.1, we have  $\langle \Lambda_k f, g_n^* \rangle_\alpha = \langle \Pi f, g_n^* \rangle_\alpha$  for any  $f \in H_\alpha$ . Similarly,  $\langle \Lambda_k f, g_* \rangle_\alpha = \langle \Pi f, g_* \rangle_\alpha$  for  $n \in \mathbb{Z}$  with  $|n| \leq k$ . The result follows.  $\square$

The processes  $S_k, U_{-k}, \dots, U_k$  in Corollary 4.2 capture at any time  $t$  the whole state of the market in the approximation model. I.e., the spot price and the forward curve are simple functions of these state variables. As we will see in Corollary 4.4 below, the forward prices of contracts with delivery periods can be expressed in these state variables as well. Note that if we assume  $\langle \Pi\beta, g_n^* \rangle, \langle \Pi\Psi, g_n^* \rangle$  are deterministic and constant, then  $(X_{-k}, \dots, X_k)$  is a  $2k+1$ -dimensional Lévy process and  $U_{-k}, \dots, U_k$  are Ornstein-Uhlenbeck processes. This corresponds to the spot price model suggested in Benth, Kallsen and Meyer-Brandis [2].

From the proof of Corollary 4.2 we find that  $S_k(0) = \langle f_k(0), g_* \rangle_\alpha$ . But then

$$S_k(0) = \langle \Lambda_k f_0, g_* \rangle_\alpha = \langle \Pi f_0, g_* \rangle_\alpha = (\Pi f_0)(0) = f_0(0).$$

Obviously,  $f_0(0)$  is equal to today's spot price, so we obtain that the starting point of the process  $S_k(t)$  in the approximation is today's spot price. Furthermore, since we have  $f_k(t, 0) = S_k(t)$  because  $g_n(0) = 0$  for all  $n \in \mathbb{Z}$ ,  $S_k(t)$  is the approximative spot price dynamics associated with  $f_k(t)$ . For  $U_n(0)$ ,  $n \in \mathbb{Z}$  invoking Corollary 3.8 shows that

$$\begin{aligned} U_n(0) &= \langle \Pi f_0, g_n^* \rangle_\alpha \\ &= \frac{1}{\sqrt{T}(1 - e^{-2\lambda T})} \int_0^T (\Pi f_0)'(y) \exp((- \lambda + \alpha/2)x) \exp\left(\frac{2\pi i}{T}nx\right) dy. \end{aligned}$$

This is the Fourier transform of the initial forward curve  $f_0$  (or, rather its derivative scaled by an exponential function). In any case, both  $S_k(0)$  and  $U_n(0)$  are given by (functionals of) the initial forward curve  $f_0$ .

Next, we would like to identify the convergence speed of our approximation, that is, the rate for the convergence in part (1) of Theorem 4.1.

**Proposition 4.3.** *Assume that  $x \mapsto f(t, x)$  is twice continuously differentiable and let  $f_k$  be the mild solution of the SPDE*

$$df_k(t) = \partial_x f_k(t) dt + \Lambda_k \beta(t) dt + \Lambda_k \Psi(t) dL(t), \quad t \geq 0, f_k(0) = \Lambda_k f_0.$$

Then, we have

$$\mathbb{E} \left[ \sup_{x \in [0, T-t]} |f_k(t, x) - f(t, x)|^2 \right] \leq \frac{A(t)}{k},$$

for any  $k > 1$ , where

$$\begin{aligned} A(t) &:= \frac{3T(1 + \alpha^{-1})}{(1 - e^{-2\lambda T})} \left\{ \|\Pi f_0\|_\alpha^2 + \int_0^T \mathbb{E}[\text{Tr}(\Psi(s)Q\Psi^*(s))] ds + \left( \int_0^T \mathbb{E}[\|\beta(s)\|_\alpha] ds \right)^2 \right\} \\ &\quad + \frac{3(1 + \alpha^{-1})}{\pi^2(1 - e^{-2\lambda T})^3} \left\{ T \mathbb{E} [|\partial_x f_\Pi(t, T) e^{T(-\lambda + \alpha/2)} - \partial_x f_\Pi(t, 0)|^2] \right. \\ &\quad \left. + \left( \int_0^T \mathbb{E} [|\partial_x^2 f_\Pi(t, x)|] e^{x(-\lambda + \alpha/2)} dx \right)^2 \right\}. \end{aligned}$$

*Proof.* In the proof of Theorem 4.1 we have shown that

$$f_k(t) = \Pi_k(f_\Pi(t)) - \mathcal{C}_{k,t} \Pi f_0 - \int_0^t \mathcal{C}_{k,t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s)),$$

where  $f_\Pi(t) := \mathcal{U}_t \Pi f_0 + \int_0^t \mathcal{U}_{t-s} (\Pi \beta(s) ds + \Pi \Psi(s) dL(s))$  for any  $t \geq 0$ . By Proposition 3.9 we have

$$\|f_\Pi(t) - \Pi_k(f_\Pi(t))\|_\alpha^2 \leq \frac{C_1(t)}{k}$$

where  $C_1(t)$  is a random variable defined by

$$C_1(t) = \frac{T |\partial_x f_\Pi(t, T) e^{T(-\lambda + \alpha/2)} - \partial_x f_\Pi(t, 0)|^2 + \left( \int_0^T |\partial_x^2 f_\Pi(t, x)| e^{x(-\lambda + \alpha/2)} dx \right)^2}{\pi^2(1 - e^{-2\lambda T})^3}.$$

Remark that from the proof of Theorem 4.1 we find for any  $h \in H_\alpha^T$

$$\|\mathcal{C}_{k,t} h\|_\alpha^2 = \|\langle h, c_{k,t} \rangle_\alpha g_*\|_\alpha^2 = |\langle h, c_{k,t} \rangle_\alpha|^2 \leq \|h\|_\alpha^2 \|c_{k,t}\|_\alpha^2,$$

and therefore, from Lemma 3.10

$$\|\mathcal{C}_{k,t}h\|_\alpha^2 \leq \|h\|_\alpha^2 \frac{C_2}{k},$$

for the constant  $C_2 = T/\pi^2(1 - e^{-2\lambda T})$ . Then, we have

$$\begin{aligned} \|f_k(t) - f_\Pi(t)\|_\alpha^2 &\leq 3\|\Pi_k(f_\Pi(t)) - f_\Pi(t)\|_\alpha^2 + 3\|\mathcal{C}_{k,t}\Pi f_0\|_\alpha^2 \\ &\quad + 3\left\|\int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))\right\|_\alpha^2 \\ &\leq \frac{3C_1(t)}{k} + \frac{3C_2}{k}\|\Pi f_0\|_\alpha^2 \\ &\quad + 3\left\|\int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))\right\|_\alpha^2. \end{aligned}$$

By Lemma 3.2 in Benth and Krühner [3], the supremum norm is bounded by the  $H_\alpha$ -norm with a constant  $c = \sqrt{1 + \alpha^{-1}}$ . Hence, taking expectations, yield

$$\begin{aligned} \mathbb{E} \left[ \sup_{x \in [0, T-t]} |f_k(t, x) - f(t, x)|^2 \right] &\leq c^2 \mathbb{E} [\|f_k(t) - f_\Pi(t)\|_\alpha^2] \\ &\leq \frac{3c^2}{k} (\mathbb{E}[C_1(t)] + C_2 \|\Pi f_0\|_\alpha^2) \\ &\quad + \frac{3c^2}{k} C_2 \left( \int_0^T \mathbb{E}[\text{Tr}(\Psi(s)Q\Psi^*(s))]ds + \left( \int_0^T \mathbb{E}[\|\beta(s)\|_\alpha] ds \right)^2 \right). \end{aligned}$$

The result follows.  $\square$

In electricity and gas markets forward contracts deliver over a future period rather than at a fixed time. The holder of the forward contract receives a uniform stream of electricity or gas over an agreed time period  $[T_1, T_2]$ . The forward prices of delivery period contracts can be derived from a "fixed-delivery time" forward curve model (see Benth et al. [5]) by

$$F(t, T_1, T_2) := \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, s - t) ds \quad (13)$$

where  $f$  is given by the SPDE (1). The following Corollary adapts Theorem 4.1 to the case of forward contracts with delivery period.

**Corollary 4.4.** *Assume the conditions of Theorem 4.1 and define*

$$F_k(t, T_1, T_2) := \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f_k(t, s - t) ds$$

for any  $0 \leq t \leq T_1 \leq T_2 \leq T$ . Then, we have

$$F_k(t, T_1, T_2) \rightarrow F(t, T_1, T_2)$$

for  $k \rightarrow \infty$  in  $L^2(\Omega)$  where  $F$  is given in (13). Furthermore,

$$F_k(t, T_1, T_2) = S_k(t) + \sum_{n=-k}^k G_n(t, T_1, T_2) \left( e^{\lambda_n t} \langle g_n^*, f_k(0) \rangle_\alpha + \int_0^t e^{\lambda_n(t-s)} dX_n(s) \right),$$

for any  $t \leq T_1 \leq T_2 \leq T$  where  $S_k(t) = \delta_0(f_k(t))$ ,

$$G_n(t, T_1, T_2) = \frac{\exp(\lambda_n(T_2 - t)) - \exp(\lambda_n(T_1 - t)) - \lambda_n(T_2 - T_1)}{\lambda_n^2 \sqrt{T}(T_2 - T_1)}$$

and  $X_n(t) := \int_0^t \langle \Pi\beta(s)ds + \Pi\Psi(s)dL(s), g_n^* \rangle_\alpha$ .

*Proof.* Theorem 4.1 yields uniform  $L^2$  convergence of the integrands appearing in  $F_k$  to the integrand appearing in  $F$  and hence the convergence follows. The representation of  $F_k$  follows immediately from part (3) of Theorem 4.1.  $\square$

We remark in passing that temperature derivatives market (see e.g. Benth and Šaltytė Benth [6]) trades in forwards with a "delivery period" as well. In this market, the forward is cash-settled against an index of the daily average temperature measured in a city over a given period.

## 5. REFINEMENT TO MARKOVIAN FORWARD PRICE MODELS

In this Section we refine our analysis to Markovian forward price models, making the additional assumption that the coefficients  $\beta$  and  $\Psi$  depend on the state of the forward curve. More specifically, we assume that

$$\beta(t) = b(t, f(t)), \tag{14}$$

$$\Psi(t) = \psi(t, f(t)), \tag{15}$$

where  $b : \mathbb{R}_+ \times H_\alpha \rightarrow H_\alpha$ ,  $\psi : \mathbb{R}_+ \times H_\alpha \rightarrow L(H_\alpha)$  are measurable Lipschitz-continuous functions of linear growth in the sense

$$\|b(t, f) - b(t, g)\|_\alpha \leq C_b \|f - g\|_\alpha, \tag{16}$$

$$\|(\psi(t, f) - \psi(t, g))\mathcal{Q}^{1/2}\|_{\text{HS}} \leq C_\psi \|f - g\|_\alpha, \tag{17}$$

and

$$\|b(t, f)\|_\alpha \leq C_b(1 + \|f\|_\alpha), \tag{18}$$

$$\|\psi(t, f)\mathcal{Q}^{1/2}\|_{\text{HS}} \leq C_\psi(1 + \|f\|_\alpha), \tag{19}$$

for positive constants  $C_b, C_\psi$ . Under these conditions there exists a unique mild solution  $f$  of the semilinear SPDE

$$df(t) = (\partial_x f(t) + b(t, f(t)))dt + \psi(t, f(t-))dL(t), \quad f(0) = f_0. \tag{20}$$

We would like to note that semilinear SPDEs are treated in the book by Peszat and Zabczyk [22] and in Tappe [25]. Additionally, we assume that

$$b(t, h) = b(t, g), \tag{21}$$

$$\psi(t, h) = \psi(t, g), \tag{22}$$

for any  $h, g \in H_\alpha$  such that  $h(x) = g(x)$  for any  $x \in [0, T - t]$ , i.e. the structure of the curve beyond our time horizon  $T$  does not influence the dynamics of the curve-valued process  $f(t)$ .

Before continuing our analysis of the arbitrage-free approximation in the Markovian case, we show a couple of useful lemmas. The first states a version of Doob's  $L^2$  inequality for Volterra-like Hilbert space-valued stochastic integrals with respect to the Lévy process  $L$ , and is essentially collected from Filipović, Tappe and Teichmann [17].

**Lemma 5.1.** *Suppose that  $\Phi \in \mathcal{L}_L^2(H_\alpha)$ . Then,*

$$\mathbb{E} \left[ \sup_{s \in [0, t]} \left\| \int_0^s \mathcal{U}_{s-r} \Phi(r) dL(r) \right\|_\alpha^2 \right] \leq 4c_t^2 \int_0^t \mathbb{E} [\|\Phi(r) \mathcal{Q}^{1/2}\|_{HS}^2] dr,$$

for  $c_t > 0$  being at most exponentially growing in  $t$ .

*Proof.* Note first that due to Benth and Krühner [3, Lemma 3.5] the  $C_0$ -semigroup  $(\mathcal{U}_t)_{t \geq 0}$  is pseudo-contractive. Filipović, Tappe and Teichmann [17, Prop. 8.7] state that there is a Hilbert space extension  $H$  of  $H_\alpha$  (i.e.  $H$  is a Hilbert space and  $H_\alpha$  is its subspace and the norm of  $H_\alpha$  equals the norm of  $H$  restricted to  $H_\alpha$ ) and a  $C_0$ -group  $(\mathcal{V}_t)_{t \in \mathbb{R}}$  on  $H$  such that  $\mathcal{V}_t|_{H_\alpha} = \mathcal{U}_t$  for  $t \geq 0$ . Then, we have

$$\begin{aligned} \sup_{s \in [0, t]} \left\| \int_0^s \mathcal{U}_{s-r} \Phi(r) dL(r) \right\|_\alpha &\leq \sup_{s \in [0, t]} \|\mathcal{V}_{s-t}\|_{\text{op}} \left\| \int_0^s \mathcal{U}_{t-r} \Phi(r) dL(r) \right\|_\alpha \\ &\leq \sup_{s \in [0, t]} \|\mathcal{V}_s\|_{\text{op}} \sup_{s \in [0, t]} \left\| \int_0^s \mathcal{U}_{t-r} \Phi(r) dL(r) \right\|_\alpha. \end{aligned}$$

Thus, by Doob's maximal inequality, Thm. 2.2.7 in Prevot and Röckner [23], we find

$$\begin{aligned} \mathbb{E} \left[ \sup_{s \in [0, t]} \left\| \int_0^s \mathcal{U}_{s-r} \Phi(r) dL(r) \right\|_\alpha^2 \right] &\leq \sup_{s \in [0, t]} \|\mathcal{V}_s\|_{\text{op}}^2 \mathbb{E} \left[ \sup_{s \in [0, t]} \left\| \int_0^s \mathcal{U}_{t-r} \Phi(r) dL(r) \right\|_\alpha^2 \right] \\ &\leq 4 \sup_{s \in [0, t]} \|\mathcal{V}_s\|_{\text{op}}^2 \mathbb{E} \left[ \left\| \int_0^t \mathcal{U}_{t-r} \Phi(r) dL(r) \right\|_\alpha^2 \right] \\ &= 4 \sup_{s \in [0, t]} \|\mathcal{V}_s\|_{\text{op}}^2 \int_0^t \mathbb{E} [\|\mathcal{U}_{t-r} \Phi(r) \mathcal{Q}^{1/2}\|_{HS}^2] dr \\ &\leq 4 \sup_{s \in [0, t]} \|\mathcal{V}_s\|_{\text{op}}^2 \sup_{s \in [0, t]} \|\mathcal{U}_s\|_{\text{op}}^2 \int_0^t \mathbb{E} [\|\Phi(r) \mathcal{Q}^{1/2}\|_{HS}^2] dr \end{aligned}$$

This proves the Lemma by letting  $c_t = \sup_{s \in [0, t]} \|\mathcal{V}_s\|_{\text{op}} \sup_{0 \leq s \leq t} \|\mathcal{U}_s\|_{\text{op}}$  and recalling that any  $C_0$ -group is bounded in operator norm by an exponentially increasing function in  $t$ . Hence,  $c_t \leq c \exp(wt)$  for some constants  $c, w > 0$ .  $\square$

We remark in passing that the above result holds for any pseudo-contractive semigroup  $\mathcal{S}_t, t \geq 0$ .

The next lemma is a useful technical result on the distance between processes and the fixed point of an integral operator defined via the mild solution of (20). The lemma plays a crucial role in showing that certain arbitrage-free approximations of (20) converge to the right limit.

**Lemma 5.2.** *For an  $H_\alpha$ -valued adapted and càdlàg stochastic process  $h$ , define*

$$V(h)(t) := \mathcal{U}_t f_0 + \int_0^t \mathcal{U}_{t-s} b(s, h(s)) ds + \int_0^t \mathcal{U}_{t-s} \psi(s, h(s-)) dL(s),$$

*for any  $t \geq 0$ . Then,  $V$  has a fixed point  $\hat{f}$  and it holds*

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|h(s) - \hat{f}(s)\|_\alpha^2 \right] \leq \frac{\pi^2}{6} \exp(4C_t) \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|V(h)(s) - h(s)\|_\alpha^2 \right],$$

*for any  $t \geq 0$  and any  $H_\alpha$ -valued adapted càdlàg stochastic processes  $h$ , with  $C_t$  being a positive constant depending on  $t$ .*

*Proof.* If  $h$  is an adapted càdlàg  $H_\alpha$ -valued stochastic process such that  $\mathbb{E}[\int_0^t \|h(s)\|_\alpha^2 ds] < \infty$ , then from the linear growth assumption (18) on  $b$  we find

$$\begin{aligned} \mathbb{E} \left[ \int_0^t \|\mathcal{U}_{t-s} b(s, h(s))\|_\alpha ds \right] &\leq C_b e^{wt} (t + \mathbb{E} \left[ \int_0^t \|h(s)\|_\alpha ds \right]) \\ &\leq C_b e^{wt} (t + \sqrt{t} \mathbb{E} \left[ \int_0^t \|h(s)\|_\alpha^2 ds \right]^{1/2}) \\ &< \infty. \end{aligned}$$

Furthermore, from the linear growth condition (19) on  $\psi$

$$\mathbb{E} \left[ \int_0^t \|\mathcal{U}_{t-s} \psi(s, h(s))\|_\alpha^2 ds \right] \leq 2C_\psi^2 e^{2wt} \left( t + \mathbb{E} \left[ \int_0^t \|h(s)\|_\alpha^2 ds \right] \right) < \infty.$$

Hence,  $V(h)$  is well-defined, and it is an adapted càdlàg process. By a straightforward estimation using again the linear growth of  $b$  and  $\psi$ , we find similarly that

$$\mathbb{E} \left[ \int_0^t \|V(h)(s)\|_\alpha^2 ds \right] \leq C_t \left( 1 + \mathbb{E} \left[ \int_0^t \|h\|_\alpha^2 ds \right] \right) < \infty,$$

for some constant  $C_t > 0$ . Therefore,  $V$  maps into its own domain and, thus, can be iterated.

We note that by general theory, the SPDE

$$df(t) = \partial_x f(t) dt + b(t, f(t)) dt + \psi(t, f(t-)) dL(t)$$

has a unique mild solution  $\hat{f}$  which has a càdlàg modification, cf. Tappe [25, Theorem 4.5, Remark 4.6]. By definition of mild solutions, we see that  $\hat{f}$  is a fix point for  $V$ , i.e.,  $V(\hat{f}) = \hat{f}$ .

Let  $g, h$  be  $H_\alpha$ -valued adapted càdlàg stochastic processes and  $t \geq 0$ . Then, we have

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|V(h)(s) - V(g)(s)\|_\alpha^2 \right] \\ \leq 2\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s \mathcal{U}_{s-r} (b(r, h(r)) - b(r, g(r))) \, dr \right\|_\alpha^2 \right] \\ + 2\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s \mathcal{U}_{s-r} (\psi(r, h(r-)) - \psi(r, g(r-))) \, dL(r) \right\|_\alpha^2 \right]. \end{aligned}$$

Consider the first term on the right hand side of the inequality. By the norm inequality for Bochner integrals and Lipschitz continuity of  $b$  in (16), we find

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s \mathcal{U}_{s-r} (b(r, h(r)) - b(r, g(r))) \, dr \right\|_\alpha^2 \right] \\ \leq \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left( \int_0^s \|\mathcal{U}_{s-r}\|_{\text{op}} \|b(r, h(r)) - b(r, g(r))\|_\alpha \, dr \right)^2 \right] \\ \leq t \mathbb{E} \left[ \sup_{0 \leq s \leq t} \int_0^s \|\mathcal{U}_{s-r}\|_{\text{op}}^2 \|b(r, h(r)) - b(r, g(r))\|_\alpha^2 \, dr \right] \\ \leq t^2 \sup_{0 \leq s \leq t} \|\mathcal{U}_s\|_{\text{op}}^2 \mathbb{E} \left[ \int_0^t \|b(r, h(r)) - b(r, g(r))\|_\alpha^2 \, dr \right] \\ \leq t^2 C_b^2 \sup_{0 \leq s \leq t} \|\mathcal{U}_s\|_{\text{op}}^2 \int_0^t \mathbb{E} [\|h(r) - g(r)\|_\alpha^2] \, dr, \end{aligned}$$

where we have applied Cauchy-Schwartz' inequality. Recall that since  $\mathcal{U}_t$  is a pseudo-contractive semigroup, we find for some  $w > 0$ , it holds that  $\sup_{0 \leq s \leq t} \|\mathcal{U}_s\|_{\text{op}}^2 \leq \exp(2wt) < \infty$ .

For the second term, we find by appealing to Lemma 5.1 and the Lipschitz continuity in (17) of  $\psi$ ,

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s \mathcal{U}_{s-r} (\psi(r, h(r-)) - \psi(r, g(r-))) \, dL(r) \right\|_\alpha^2 \right] \\ \leq 4c_t^2 \int_0^t \mathbb{E} [\|(\psi(r, h(r)) - \psi(r, g(r))) \mathcal{Q}^{1/2}\|_{\text{HS}}^2] \, dr \\ \leq 4c_t^2 C_\psi^2 \int_0^t \mathbb{E} [\|h(r) - g(r)\|_\alpha^2] \, dr \end{aligned}$$

Here, the constant  $c_t$  is from Lemma 5.1. Denote by  $C_t$  the constant

$$C_t := 2C_b^2 t^2 \sup_{s \in [0, t]} \|\mathcal{U}_s\|_{\text{op}} + 8c_t^2 C_\psi^2 t.$$

Then, we have

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|V^n(h)(s) - V^n(g)(s)\|_\alpha^2 \right] \\
& \leq C_t \int_0^t \mathbb{E} [\|V^{n-1}(h)(s_1) - V^{n-1}(g)(s_1)\|_\alpha^2] ds_1 \\
& \leq C_t^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \mathbb{E} [\|h(s_n) - g(s_n)\|_\alpha^2] ds_n \cdots ds_1 \\
& \leq \frac{C_t^n}{n!} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|h(s) - g(s)\|_\alpha^2 \right],
\end{aligned}$$

for any  $n \in \mathbb{N}$ . Denote by  $L_a^2(\Omega, D([0, t], H_\alpha))$  the space of  $H_\alpha$ -valued adapted càdlàg stochastic processes  $\{f(s)\}_{s \in [0, t]}$  for which  $\mathbb{E}[\sup_{s \in [0, t]} \|f(s)\|_\alpha^2] < \infty$ . Equip this space with the norm  $\|\cdot\|_t$  defined by

$$\|f\|_t^2 := \mathbb{E} \left[ \sup_{s \in [0, t]} \|f(s)\|_\alpha^2 \right]$$

for  $f \in L_a^2(\Omega, D([0, t], H_\alpha))$ . From the estimation above, we see that  $V$  operates on the normed space  $L_a^2(\Omega, D([0, t], H_\alpha))$ . Moreover,  $V^n$  is Lipschitz continuous with constant strictly less than 1 for  $n$  sufficiently large. Thus, by Banach's fixed point theorem there is at most one fixed point for  $V$ . Hence,  $\hat{f}$  is the unique fix point for  $V$ . Furthermore, we have

$$\begin{aligned}
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|V^n(h)(s) - h(s)\|_\alpha^2 \right]^{1/2} & \leq \sum_{k=0}^{n-1} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|V^{k+1}(h)(s) - V^k(h)(s)\|_\alpha^2 \right]^{1/2} \\
& \leq \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|V(h)(s) - h(s)\|_\alpha^2 \right]^{1/2} \sum_{k=0}^{n-1} \left( \frac{C_t^k}{k!} \right)^{1/2}.
\end{aligned}$$

From Cauchy-Schwartz' inequality and we have that

$$\begin{aligned}
\sum_{k=0}^{n-1} \left( \frac{C_t^k}{k!} \right)^{1/2} & = \sum_{k=0}^{n-1} (k+1)^{-1} \left( \frac{(k+1)^2 C_t^k}{k!} \right)^{1/2} \\
& \leq \left( \sum_{k=0}^{n-1} \frac{1}{(k+1)^2} \right)^{1/2} \left( \sum_{k=0}^{n-1} \frac{(k+1)^2 C_t^k}{k!} \right)^{1/2} \\
& \leq \frac{\pi}{\sqrt{6}} \left( \sum_{k=0}^{n-1} \frac{4^k C_t^k}{k!} \right)^{1/2} \\
& \leq \frac{\pi}{\sqrt{6}} \exp(2C_t),
\end{aligned}$$

where we have used the elementary inequality  $k+1 \leq 2^k$ ,  $k \in \mathbb{N}$ .  $\square$

Let us define the Lipschitz continuous functions  $b_\Pi := \Pi \circ b$  and  $\psi_\Pi := \Pi \circ \psi$ . Then, Tappe [25, Theorem 4.5] yields a mild solution  $f_\Pi$  for the SPDE

$$df_\Pi(t) = (\partial_x f_\Pi(t) + b_\Pi(t, f_\Pi(t))) dt + \psi_\Pi(t, f_\Pi(t-)) dL(t), \quad f_\Pi(0) = \Pi f_0. \quad (23)$$



Furthermore, it will be convenient to use the notations

$$b_k(t, h) := \Lambda_k(b(t, h)), \quad (24)$$

$$\psi_k(t, h) := \Lambda_k(\psi(t, h)) \quad (25)$$

for any  $h \in H_\alpha$ ,  $t \geq 0$ .

In the proof of Theorem 4.1 we compared the solution  $f$  to the projected solution  $\Pi f$  which are essentially the same due to properties of  $\Pi$ . Then we compared  $\Pi f$  to  $f_\Pi$  which again had been essentially the same. Finally, we compared  $\Pi_k f_\Pi$  to solutions of the projected SPDE where the difference was given by a certain Lie-commutator. However, in the Markovian setting we want to change the dependencies of the coefficients as well, which complicates the proof of the approximation result.

**Theorem 5.3.** *Denote by  $\hat{f}_k$  be the mild solution of the SPDE*

$$d\hat{f}_k(t) = (\partial_x \hat{f}_k(t) + b_k(t, \hat{f}_k(t))) dt + \psi_k(t, \hat{f}_k(t-)) dL(t), \quad \hat{f}_k(0) = \Lambda_k f_0, t \geq 0.$$

*Then,  $\hat{f}_k \in H_\alpha^{T,k}$  is a strong solution, and we have*

$$\mathbb{E} \left[ \sup_{t \in [0, T], x \in [0, T-t]} |\hat{f}_k(t, x) - f(t, x)|^2 \right] \rightarrow 0$$

for  $k \rightarrow \infty$ .

*Proof.* First we note that a unique mild solution  $\hat{f}_k$  of the SPDE exists due to Tappe [25, Theorem 4.5]. Define

$$V_k(h)(t) := \mathcal{U}_t f_k(0) + \int_0^t \mathcal{U}_{t-s} (b_k(s, h(s)) ds + \psi_k(s, h(s-)) dL(s)),$$

for any  $k \in \mathbb{N}$ ,  $t \geq 0$  and any adapted càdlàg stochastic process  $h$  in  $H_\alpha$ . Let  $f_k$  be defined as

$$\begin{aligned} f_k(t) &:= \mathcal{U}_t f_k(0) + \int_0^t \mathcal{U}_{t-s} (b_k(s, f(s)) ds + \psi_k(s, f(s)) dL(s)) \\ &= \mathcal{U}_t f_k(0) + \int_0^t \mathcal{U}_{t-s} (b_k(s, f_\Pi(s)) ds + \psi_k(s, f_\Pi(s-)) dL(s)) \\ &= V_k(f_\Pi)(t), \end{aligned}$$

for  $f_k(0) = \Lambda_k f(0)$ . Moreover,  $\hat{f}_k(t) = V_k(\hat{f}_k)(t)$ . By Lemma 5.2, it holds

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|f_\Pi(t) - \hat{f}_k(t)\|_\alpha^2 \right] \leq \frac{\pi^2}{6} \exp(4C_t) \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|f_k(s) - f_\Pi(s)\|_\alpha^2 \right],$$

for any  $k \in \mathbb{N}$ ,  $t \geq 0$  and  $C_t$  given in the lemma (recall from Section 2 that the operator norm of the shift semigroup  $\mathcal{U}_t$  is uniformly bounded by the constant  $C_\mathcal{U}$ ). By the definition of  $f_k$  and  $f_\Pi$  we find

$$\begin{aligned} \|f_k(s) - f_\Pi(s)\|_\alpha^2 &\leq 2 \left\| \int_0^s \mathcal{U}_{s-r} (b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r))) dr \right\|_\alpha^2 \\ &\quad + 2 \left\| \int_0^s \mathcal{U}_{s-r} (\psi_k(r, f_\Pi(r-)) - \psi_\Pi(r, f_\Pi(r-))) dL(r) \right\|_\alpha^2. \end{aligned}$$

Consider the first term on the right-hand side of the inequality. By the norm inequality for Bochner integrals, Cauchy-Schwartz' inequality and boundedness of the operator norm of  $\mathcal{U}_t$  we find (for  $s \leq t$ )

$$\begin{aligned}
& \left\| \int_0^s \mathcal{U}_{s-r}(b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r))) dr \right\|_\alpha^2 \\
& \leq \left( \int_0^s \|\mathcal{U}_{s-r}(b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r)))\|_\alpha dr \right)^2 \\
& \leq t \int_0^t \|\mathcal{U}_{s-r}(b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r)))\|_\alpha^2 dr \\
& \leq tC_{\mathcal{U}}^2 \int_0^t \|b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r))\|_\alpha^2 dr \\
& \leq tC_{\mathcal{U}}^2 \int_0^t \|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2 dr
\end{aligned}$$

Here,  $\mathcal{I}$  denotes the identity operator on  $H_\alpha^T$ . Hence, using Lemma 5.1 and the fact that  $\{\mathcal{U}\}_{t \geq 0}$  is pseudo-contractive,

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|f_k(s) - f_\Pi(s)\|_\alpha^2 \right] \\
& \leq 2tC_{\mathcal{U}}^2 \int_0^t \mathbb{E} [\|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2] dr \\
& \quad + 2\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left\| \int_0^s \mathcal{U}_{s-r}(\psi_k(r, f_\Pi(r-)) - \psi_\Pi(r, f_\Pi(r-))) dL(r) \right\|_\alpha^2 \right] \\
& \leq 2tC_{\mathcal{U}}^2 \int_0^t \mathbb{E} [\|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2] dr \\
& \quad + 8c_t^2 \int_0^t \mathbb{E} [\|(\psi_k(r, f_\Pi(r)) - \psi_\Pi(r, f_\Pi(r)))\mathcal{Q}^{1/2}\|_{\text{HS}}^2] dr \\
& \leq 2tC_{\mathcal{U}}^2 \int_0^t \mathbb{E} [\|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2] dr \\
& \quad + 8c_t^2 \int_0^t \mathbb{E} [\|(\Pi_k - \mathcal{I})\psi_\Pi(r, f_\Pi(r))\mathcal{Q}^{1/2}\|_{\text{HS}}^2] dr.
\end{aligned}$$

Denote by

$$\begin{aligned}
K_t(k) &:= 2tC_{\mathcal{U}}^2 \int_0^t \mathbb{E} [\|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2] dr \\
& \quad + 8c_t^2 \int_0^t \mathbb{E} [\|(\Pi_k - \mathcal{I})\psi_\Pi(r, f_\Pi(r))\mathcal{Q}^{1/2}\|_{\text{HS}}^2] dr,
\end{aligned}$$

for  $k \in \mathbb{N}$ . By standard norm inequalities, we have

$$\begin{aligned} K_t(k) &:= 4tC_{\mathcal{U}}^2(1 + \|\Pi_k\|_{\text{op}}^2) \int_0^t \mathbb{E} [\|b_{\Pi}(r, f_{\Pi}(r))\|_{\alpha}^2] dr \\ &\quad + 16c_t^2(1 + \|\Pi_k\|_{\text{op}}^2) \int_0^t \mathbb{E} [\|\psi_{\Pi}(r, f_{\Pi}(r))\|_{\text{op}}^2] dr, \end{aligned}$$

which is seen to be bounded uniformly in  $k \in \mathbb{N}$  from Proposition 3.6. Hence, we have  $K_t(k) \rightarrow 0$  for  $k \rightarrow \infty$  and any  $t \geq 0$  by the dominated convergence theorem because  $(\Pi_k - \mathcal{I})h \rightarrow 0$  for  $k \rightarrow \infty$  and any  $h \in H_{\alpha}^T$ . Thus, we find

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|f_k(t) - \hat{f}_k(t)\|_{\alpha}^2 \right] \rightarrow 0,$$

for  $k \rightarrow \infty$ . Finally,  $f_{\Pi}(t, x) = f(t, x)$  for any  $t \in [0, T]$ ,  $x \in [0, T - t]$ . Moreover, from Lemma 3.2 in Benth and Krühner [3] the sup-norm is dominated by the  $H_{\alpha}$ -norm, and therefore we have

$$\mathbb{E} \left[ \sup_{t \in [0, T], x \in [T-t]} |\hat{f}_k(t, x) - f(t, x)|^2 \right] \leq c \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|\hat{f}_k(t) - f_{\Pi}(t)\|_{\alpha}^2 \right] \rightarrow 0,$$

for  $k \rightarrow \infty$ . The Proposition follows.  $\square$

The philosophy in Thm. 5.3 is to take  $f(t)$  as the actual forward curve dynamics, and study finite dimensional approximations  $\hat{f}_k(t)$  of it. By construction,  $\hat{f}_k$  solves a HJMM dynamics which yields that the approximating forward curves become arbitrage-free. From the main theorem, the approximations  $\hat{f}_k(t)$  converge uniformly to  $f(t)$  for  $x \in [0, T - t]$ . As time  $t$  progresses, the times to maturity  $x \geq 0$  for which we obtain convergence shrink. The reason is that information of  $f$  is transported to the left in the dynamics of the SPDE. We recall that the approximation of  $f$  is constructed by first localizing  $f$  to  $x \in [0, T]$  for a fixed time horizon  $T$  by the projection operator  $\Pi$  down to  $H_{\alpha}^T$ , and next creating finite-dimensional approximations of this.

Alternatively, we may use  $f_{\Pi}(t)$  as our forward price model. Then, the finite dimensional approximation  $f_k(t)$  will converge uniformly over all  $x \in [0, T]$ . In practice, there will be a time horizon for the futures market for which we have no information. For example, in liberalized power markets like NordPool and EEX, there are no futures contracts traded with settlement beyond 6 years. Hence, it is a delicate task to model the dynamics of the futures price curve beyond this horizon. The alternative is then clearly to restrict the modelling perspective to the dynamics with the maturities confined in  $x \in [0, T]$ . Indeed, in such a context the structural conditions (21) and (22) will be trivially satisfied as we restrict our model parameters in any case to the behaviour on  $x \in [0, T]$ .

We end our paper with a short discussion on a possible numerical implementation of  $\hat{f}_k(t)$ , the finite-dimensional approximation of  $f(t)$ . Since  $\hat{f}_k(t) \in H_{\alpha}^{T,k}$ , we can express it as

$$\hat{f}_k(t) = \hat{f}_{k,*}(t) + \sum_{n=-k}^k g_n \hat{f}_{k,n}(t),$$

where  $\widehat{f}_{k,*}(t) = \widehat{f}_k(t, 0)g_*$  and  $\widehat{f}_{k,n}(t) = \langle \widehat{f}_k(t), g_n^* \rangle_\alpha$  are  $\mathbb{C}$ -valued functions. For any  $h \in H_\alpha^{T,k}$  it follows that  $b_k(t, h) \in H_\alpha^{T,k}$ . Define for  $n = -k, \dots, k$  the functions

$$\begin{aligned} \bar{b}_{k,n} : \mathbb{R}_+ \times \mathbb{C}^{2k+2} &\rightarrow \mathbb{C}; & (t, x_*, x_{-k}, \dots, x_k) &\mapsto \left\langle b_k(t, x_*g_* + \sum_{j=-k}^k x_j g_j), g_n^* \right\rangle_\alpha, \\ \bar{b}_{k,*} : \mathbb{R}_+ \times \mathbb{C}^{2k+2} &\rightarrow \mathbb{C}; & (t, x_*, x_{-k}, \dots, x_k) &\mapsto \left\langle b_*(t, x_*g_* + \sum_{j=-k}^k x_j g_j), g_n^* \right\rangle_\alpha. \end{aligned}$$

Furthermore,  $\psi_k(t, h) \in L_{\text{HS}}(H_\alpha, H_\alpha^{T,k})$ . Thus, for any  $g \in H_\alpha$  we have that  $\psi_k(t, h)(g) \in H_\alpha^{T,k}$ . We define the mappings

$$\begin{aligned} \bar{\psi}_{k,n} : \mathbb{R}_+ \times \mathbb{C}^{2k+2} &\rightarrow H_\alpha^*; & (t, x_*, x_{-k}, \dots, x_k) &\mapsto \left\langle \psi_k(t, x_*g_* + \sum_{j=-k}^k x_j g_j)(\cdot), g_n^* \right\rangle_\alpha \\ \bar{\psi}_{k,*} : \mathbb{R}_+ \times \mathbb{C}^{2k+2} &\rightarrow H_\alpha^*; & (t, x_*, x_{-k}, \dots, x_k) &\mapsto \left\langle \psi_*(t, x_*g_* + \sum_{j=-k}^k x_j g_j)(\cdot), g_n^* \right\rangle_\alpha \end{aligned}$$

for  $n = -k, \dots, k$ . Now, since  $\partial_x g_* = 0$  and  $\partial_x g_n = \lambda_n g_n + g_*/\sqrt{T}$ , we find from the SPDE of  $\widehat{f}_k$  the following  $2k+2$  system of stochastic differential equations (after comparing terms with respect to the Riesz basis functions),

$$\begin{aligned} d\widehat{f}_{k,*}(t) &= \left( \frac{1}{\sqrt{T}} \sum_{n=-k}^k \widehat{f}_{k,n}(t) + \bar{b}_{k,*}(t, \widehat{f}_{k,*}(t), \widehat{f}_{k,-k}(t), \dots, \widehat{f}_{k,k}(t)) \right) dt \\ &\quad + d\bar{\psi}_{k,*}(t, \widehat{f}_{k,*}(t-), \widehat{f}_{k,-k}(t-), \dots, \widehat{f}_{k,k}(t-))(L(t)) \\ d\widehat{f}_{k,-k}(t) &= \left( \lambda_{-k} \widehat{f}_{k,-k}(t) + \bar{b}_{k,-k}(t, \widehat{f}_{k,*}(t), \widehat{f}_{k,-k}(t), \dots, \widehat{f}_{k,k}(t)) \right) dt \\ &\quad + d\bar{\psi}_{k,-k}(t, \widehat{f}_{k,*}(t-), \widehat{f}_{k,-k}(t-), \dots, \widehat{f}_{k,k}(t-))(L(t)) \\ &\quad \dots \\ &\quad \dots \\ d\widehat{f}_{k,k}(t) &= \left( \lambda_k \widehat{f}_{k,k}(t) + \bar{b}_{k,k}(t, \widehat{f}_{k,*}(t), \widehat{f}_{k,-k}(t), \dots, \widehat{f}_{k,k}(t)) \right) dt \\ &\quad + d\bar{\psi}_{k,k}(t, \widehat{f}_{k,*}(t-), \widehat{f}_{k,-k}(t-), \dots, \widehat{f}_{k,k}(t-))(L(t)) \end{aligned}$$

In a compact matrix notation, defining  $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_{2k+2}(t))'$  and

$$A = \begin{bmatrix} \frac{1}{\sqrt{T}} & \frac{1}{\sqrt{T}} & \frac{1}{\sqrt{T}} & \cdots & \frac{1}{\sqrt{T}} \\ 0 & \lambda_{-k} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{-k+1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix},$$

we have the dynamics

$$d\mathbf{x}(t) = (A\mathbf{x}(t) + \bar{\mathbf{b}}_k(t, \mathbf{x}(t))) dt + d\bar{\psi}_k(t, \mathbf{x}(t-))(L(t)),$$

with  $\hat{f}_{k,*} = x_1, \hat{f}_{k,-k} = x_2, \dots, \hat{f}_{k,k} = x_k$ . Using for example an Euler approximation, we can derive an iterative numerical scheme for this stochastic differential equation in  $\mathbb{C}^{2k+2}$ . We refer to Kloeden and Platen [21] for a detailed analysis of numerical solution of stochastic differential equations driven by Wiener noise.

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